https://www.linkedin.com/feed/update/urn:li:activity:6539382415616540672
Prove the inequalities.

2580. Proposed by Hojoo Lee, student Kwangwoon University Kangwon Do South Korea

Suppose that a,b and c are positive real numbers Prove that

$$\frac{b+c}{a^2+bc}+\frac{c+a}{b^2+ca}+\frac{a+b}{c^2+ab} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Solution by Arkady Alt , San Jose ,California, USA.

$$\sum\frac{1}{a}-\sum\frac{b+c}{a^2+bc} = \sum\left(\frac{1}{a}-\frac{b+c}{a^2+bc}\right)=\sum\frac{(a-b)(a-c)}{a^3+q},$$

where $q := abc$. Due to symmetry assume that $a \ge b \ge c$. Then

$$\sum\frac{(a-b)(a-c)}{a^3+q} = \frac{(a-b)(a-c)}{a^3+q}+(b-c)\left(\frac{b-a}{b^3+q}-\frac{c-a}{c^3+q}\right)=$$

$$\frac{(a-b)(a-c)}{a^3+q}+(b-c)\left(\frac{a-c}{c^3+q}-\frac{a-b}{b^3+q}\right)\ge 0,$$ because

$\dfrac{(a-b)(a-c)}{a^3+q}\ge 0$ and $\dfrac{a-c}{c^3+q}\ge\dfrac{a-b}{c^3+q}\ge\dfrac{a-b}{b^3+q}.$

2581. Proposed by Hojoo Lee, student Kwangwoon University Kangwon Do South Korea

Suppose that a,b and c are positive real numbers Prove that

$$\frac{ab+c^2}{a+b}+\frac{bc+a^2}{b+c}+\frac{ca+b^2}{c+a}\ge a+b+c.$$

Solution by Arkady Alt , San Jose ,California, USA.

Note that $\sum\frac{ab+c^2}{a+b}\ge a+b+c \iff \sum\left(\frac{ab+c^2}{a+b}+c\right)\ge 2(a+b+c) \iff$

(**1**) $\displaystyle\sum\frac{(b+c)(c+a)}{a+b}\ge 2(a+b+c)$

Let $x := b+c,\ y := c+a,\ z := a+b$. Then (**1**) becomes

$$\sum\frac{xy}{z}\ge x+y+z \iff \sum x^2y^2\ge xyz(x+y+z) \iff \sum(xy-yz)^2\ge 0.$$

2585. Proposed by Vedula N Murty Visakhapatnam India.

Prove that for $0<\theta<\pi/2$,

$$\tan\theta+\sin\theta>2\theta.$$

Solution by Arkady Alt , San Jose ,California, USA.

Solution1 (traditional with, calculus)

Let $h(x) := \tan x+\sin x-2x$ then $h'(x)=\dfrac{1}{\cos^2x}+\cos x-2$ and since $0<\cos x<1$

we obtain $h'(x)>\dfrac{1}{\cos^2x}+\cos^2x-2=\left(\dfrac{1}{\cos x}-\cos x\right)^2\ge 0.$

Hence $h(x)$ increase in $(0,\pi/2)$ and, therefore, $h(x)>h(0)=0$.

Solution2. (elementary, without calculus)

First note that $\tan x+\sin x>4\tan\dfrac{x}{2}$ for $x\in(0,\pi/2)$

Indeed, let $t := \tan\dfrac{x}{2}$ then $t\in(0,1)$ and $\tan x+\sin x>4\tan\dfrac{x}{2}\iff$

$\dfrac{2t}{1-t^2} + \dfrac{2t}{1+t^2} > 4t \Leftrightarrow \dfrac{t}{1-t^2} + \dfrac{t}{1+t^2} > 2 \Leftrightarrow \dfrac{2}{1-t^4} > 2 \Leftrightarrow$
$\dfrac{1}{1-t^4} > 1 \Leftrightarrow t^4 > 0.$

Since $\tan\dfrac{x}{2} > \dfrac{x}{2}$ then $\tan x + \sin x > 4\tan\dfrac{x}{2} > 4 \cdot \dfrac{x}{2} = 2x.$

Remark.

If $x \in (0, \pi/2]$ then $\tan\theta + \sin\theta \geq 2\theta$ and occurs only if $x = 0.$